The Under Addressed Optical MIMO Channel: Capacity and Outage

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We study an optical space-division multiplexed system where the number of modes that are addressed by the transmitter and receiver is allowed to be smaller than the total number of optical modes supported by the fiber. This situation will be of relevance if fibers supporting more modes than can be processed with current MIMO technology are deployed with the purpose of future-proof installation. We calculate the ergodic capacity and the outage probability of the link and study their dependence on the number of addressed modes at the transmitter and receiver.

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One of the most intensely explored approaches for dealing with the imminent capacity crunch of optical communications systems [1] is the implementation of space division multiplexing (SDM) in multi-mode optical fibers [2]- [5]. With this approach, information is transmitted simultaneously over multiple spatial modes of the optical fiber, while relying on multiple-input and multiple-output (MIMO) signal processing algorithms to resolve issues related to mode coupling in the process of propagation. With today's technology, the number of fiber modes that can be effectively supported for the transmission of information is limited almost exclusively by the complexity of the MIMO algorithms and by the speed of signal-processing hardware. As these technologies continuously improve with time, one may consider deploying multi-mode fibers admitting a larger number of spatial modes than can be processed today, with the intention of harvesting the full capacity of the fiber in the future. Such a solution, initially proposed and considered by Winzer and Foschini in [3], does not come without a price. Part of the transmitted signal energy couples into fiber modes that are not detected at the receiver, thereby resulting in the reduction of achievable capacity. In what follows, we refer to the multi-mode fiber-optic channel in which not all supported modes are coupled to transmitters or receivers, as the under-addressed MIMO channel. Analysis of its performance is the prime goal of this work.

We consider a system using a total of m scalar modes (counting both spatial modes and polarizations) and where the number of modes addressed by the transmitter and receiver are m_t and m_r, respectively. We explore two distinct regimes of operation, referred to as the ergodic and the non-ergodic regimes [6]. In the ergodic regime, a single frame of the error-correcting code samples the entire channel statistics, whereas in the non-ergodic regime, the channel within each code-frame is assumed to be constant. In the fiber-optic scenario, the ergodic regime is relevant in particular when the channel correlation bandwidth is very small relative to the bandwidth of the signal, as a result of large modal dispersion [4]. It may also become relevant if schemes actively randomizing fiber mode-coupling on a time-scale much shorter than the the error-correcting code-frame are introduced. In the ergodic regime, performance is evaluated in terms of the *ergodic capacity*, which is the channel capacity averaged with respect to all channel realizations. In the non-ergodic regime, performance is characterized in terms of the system *outage probability*. We derive these quantities analytically and present their dependence on m, m_t and m_r. A particularly interesting outcome of our study is that when $m_t + m_r > m$, a throughput equivalent to $m_t + m_r - m$ decoupled single-mode channels can be guaranteed. In the fast-changing channel regime this implies that the ergodic capacity is never smaller than $(m_t + m_r - m)$ single-input single-output (SISO) channels, whereas in the non-ergodic regime a throughput equivalent to (m_t+m_r-m) SISO channels can be achieved with zero outage probability.

In the absence of sufficient experimental characterization, we adopt the description of the multi-mode fiber as a unitary system with strong mode-coupling, as was used in most previous studies [3]- [5] of the multi-mode transmission problem. By doing so, we ignore the effects of mode-dependent losses and justify the description of the overall $m \times m$ transfer matrix $\mathbf{H}$ as a random instantiation drawn uniformly from the ensemble of all $m \times m$ unitary matrices (Haar distributed). In addition, as was done in [3], we assume that the average power generated by each of the m_t transmitters is constant, regardless of the value of m_t. Under these conditions the channel can be described as

$$\underline{\mathbf{y}} = \rho \mathbf{H}_{11} \underline{\mathbf{x}} + \underline{\mathbf{z}} \ , \tag{1}$$

where the vector $\underline{\mathbf{x}}$ containing m_t complex components, represents the transmitted signal, the vector $\underline{\mathbf{y}}$ containing m_r complex components, represents the received signal, and $\underline{\mathbf{z}}$ accounts for the presence of additive Gaussian

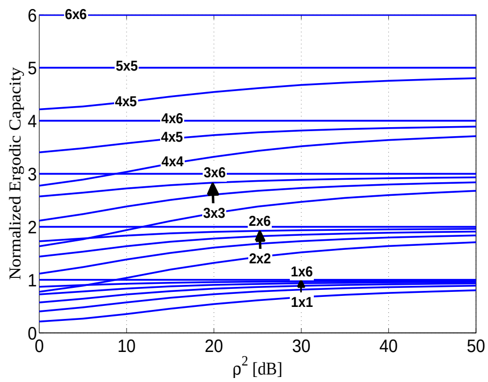

Fig. 1. The ergodic capacity normalized by $C(1, 1, 1; \rho) = \log(1 + \rho^2)$ vs. ρ^2 for various combinations of $m_t \times m_r$ with $m = 6$.

noise. The m_r components of $\underline{\mathbf{z}}$ are statistically independent, circularly symmetric complex zero-mean Gaussian variables of unit energy $\mathbb{E}(|z_j|^2) = 1$, and the components of $\underline{\mathbf{x}}$ are constrained such that the average energy of each component is equal to 1 $\mathbb{E}(|x_j|^2) = 1$. The term ρ is proportional to the optical power per excited mode so that ρ^2 is equal to the signal-to-noise ratio (SNR) in the single mode ($m = 1$) case. The matrix $\mathbf{H}_{11}$ is a block of size $m_r \times m_t$ within the $m \times m$ random unitary matrix $\mathbf{H}$

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \ .$$

As a first stage in our analysis we establish the relation between the transfer matrix $\mathbf{H}_{11}$ and the Jacobi ensemble of random matrices [7,8]. The Jacobi ensemble with (integer) parameters k_1, k_2, n (satisfying k_1, $k_2 \geq n$) is denoted $\mathcal{J}(k_1, k_2, n)$ and can be constructed as $G_1^\dagger G_1 (G_1^\dagger G_1 + G_2^\dagger G_2)^{-1}$, where G_1 and G_2 are statistically independent $k_1 \times n$ and $k_2 \times n$ random Gaussian matrices, respectively. By the term Gaussian matrix we are referring to a matrix whose entries are zero-mean complex Gaussian variables with unit-variance. The joint probability density function (PDF) of the eigenvalues of this ensemble is given in [7].

It is known [9] that for m_t, m_r satisfying the condition $m_t + m_r \leq m$, the squared non-zero singular values of $\mathbf{H}_{11}$ have the same distribution as the eigenvalues of the Jacobi ensemble $\mathcal{J}(m_{\max}, m - m_{\max}, m_{\min})$, where $m_{\min} = \min\{m_t, m_r\}$ and $m_{\max} = \max\{m_t, m_r\}$. For $m_t + m_r > m$ it can be shown [11] that $m_t + m_r - m$ singular values of $\mathbf{H}_{11}$ are equal to unity with probability 1, whereas the remaining $m - m_{\max}$ non-zero singular values of $\mathbf{H}_{11}$ are equal to the non-zero singular values of $\mathbf{H}_{22}$, and hence follow the distribution of the Jacobi ensemble $\mathcal{J}(m - m_{\min}, m_{\min}, m - m_{\max})$. The latter property can be seen by noting that the unitarity of $\mathbf{H}$ implies that $\mathbf{H}_{11}^\dagger \mathbf{H}_{11} + \mathbf{H}_{21}^\dagger \mathbf{H}_{21} = \mathrm{I}_{m_t}$ and $\mathbf{H}_{21} \mathbf{H}_{21}^\dagger + \mathbf{H}_{22} \mathbf{H}_{22}^\dagger = \mathrm{I}_{m-m_r}$.

Since the noise is additive circularly symmetric Gaussian, the capacity for a given channel realization is known [6] and given by $\log\left[\det(\mathrm{I}_{m_t} + \rho^2 \mathbf{H}_{11}^\dagger \mathbf{H}_{11})\right]$. The ergodic capacity is obtained by averaging this expression over all channel realizations $\mathbf{H}_{11}$. It can be expressed in the form

$$C(m_t, m_r, m; \rho) = \mathbb{E}[\sum_{i=1}^{m_{\min}} \log(1 + \rho^2 \lambda_i)] \ . \tag{2}$$

where the expectation is over $\lambda_1, \ldots, \lambda_{m_{\min}}$, the squared nonzero singular values of $\mathbf{H}_{11}$. In the case $m_t + m_r \leq m$, using the the joint PDF of λ_j one finds that the ergodic capacity satisfies [11]

$$C(m_t, m_r, m; \rho) = \int_0^1 \log(1 + \lambda\rho^2) \times \\ \times \lambda^\alpha (1 - \lambda)^\beta \sum_{k=0}^{m_{\min}-1} b_{k,\alpha,\beta}^{-1} [P_k^{(\alpha,\beta)}(1 - 2\lambda)]^2 d\lambda \ , \tag{3}$$

where $P_k^{(\alpha,\beta)}(x)$ are the Jacobi polynomials (see [10, 8.96]), $\alpha = |m_r - m_t|$, $\beta = m - m_t - m_r$, and the coefficients $b_{k,\alpha,\beta}$ are given by

$$b_{k,\alpha,\beta} = \frac{1}{2k + \alpha + \beta + 1} \binom{2k + \alpha + \beta}{k} \binom{2k + \alpha + \beta}{k + \alpha}^{-1} .$$

Note that the above result, as well as all results in what follows are symmetric with respect to m_t and m_r, which is a consequence of our choice to maintain the power per transmitter constant in all cases.

To obtain the ergodic capacity in the case where $m_t + m_r > m$, we use the distribution of the singular values in that case, turning Eq. (2) into

$$C(m_t, m_r, m; \rho) = (m_t + m_r - m) C(1, 1, 1; \rho) + \\ + C(m - m_r, m - m_t, m; \rho) \ , \tag{4}$$

where $C(1, 1, 1; \rho) = \log(1 + \rho^2)$ and where $C(m - m_r, m - m_t, m; \rho)$ is given by Eq. (3). Note that the second term on the right-hand-side of (4) reduces to 0 when m_t, or m_r is equal to m. The form of Eq. (4) follows from the fact that $m_t + m_r - m$ of the singular values are equal to unity, while the remaining singular values correspond to the Jacobi ensemble, as discussed earlier. In Fig. 1 we show an example where the ergodic capacity of the channel is plotted as a function of ρ^2 for the case of $m = 6$. The ergodic capacity in the figure was normalized to $\log(1 + \rho^2)$ (which is also the capacity in the single-mode case).

We now turn to the analysis of the non-ergodic case, in which the channel realization is assumed to be constant within each given code-frame. Traditional optical communications systems, which are implemented over single-mode optical fibers ($m = 2$), usually operate in this regime. As discussed earlier, the figure of merit in this regime is the outage probability P_{out}, defined as the probability that the capacity induced by the channel realization is lower than the rate R at which the link is chosen to operate. Note that we assume that the chan-

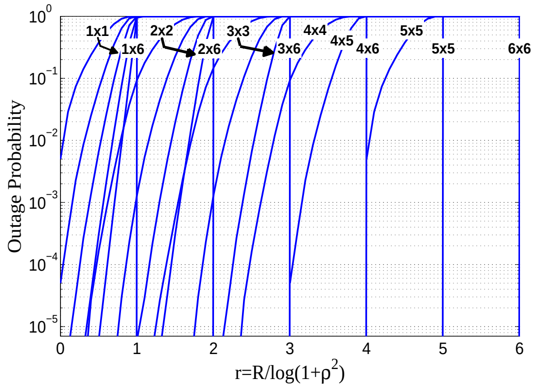

Fig. 2. Outage probability versus the normalized rate r. The number of supported modes is $m = 6$ and $\rho^2 =$ 20dB.

nel instantiation is unknown to the transmitter, thus it can not adapt the transmission rate. The outage probability can be formally expressed as

$$P_{out}(m_t, m_r, m; R) = \\ = Pr\left[\log\det(\mathrm{I}_{m_t} + \rho^2 \mathbf{H}_{11}^{\dagger}\mathbf{H}_{11}) < R\right] . \quad (5)$$

For $m_t + m_r \leq m$ Eq. (5) can be readily evaluated by using once again the distribution of the eigenvalues of the Jacobbi ensemble [7]. Defining a parameter r such that the system transmission rate is given by $R = r\log(1 + \rho^2)$, we obtain

$$P_{out}(m_t, m_r, m; R) = K^{-1}_{m_t,m_r,m} \int_{\mathcal{B}} \prod_{i=1}^{m_{\min}} \lambda_i^{|m_r - m_t|} \times \\ \times (1-\lambda_i)^{m-m_r-m_t} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda \; , \quad (6)$$

where $K_{m_t,m_r,m}$ is a normalization factor and where $\mathcal{B}$, the range of integration includes all ordered singular values of $\mathbf{H}_{11}$, $0 \leq \lambda_1 \leq \ldots \leq \lambda_{m_{\min}} \leq 1$, for which $\prod_{i=1}^{m_{\min}}(1 + \rho^2\lambda_i) < (1 + \rho^2)^r$.

For the case of $m_t + m_r > m$, since $m_t + m_r - m$ eigenvalues of $\mathbf{H}_{11}^{\dagger}\mathbf{H}_{11}$ are unity and the other $m - m_{\max}$ nonzero eigenvalues are equal to the non-zeros eigenvalues of $\mathbf{H}_{22}^{\dagger}\mathbf{H}_{22}$, it can be shown [11] that

$$P_{out}(m_t, m_r, m; r\log(1+\rho^2)) = \\ = P_{out}(m - m_r, m - m_t, m; \tilde{r}\log(1+\rho^2)) \; , \quad (7)$$

where $\tilde{r}$ is the larger between $r - (m_t + m_r - m)$ and 0, and where the right-hand-side drops to 0, when m_r, or m_t equals m. Equation (7) implies that in the case $m_t + m_r > m$ the outage probability is identical to that of a channel with $m - m_r$ modes addressed by the transmitter and $m - m_t$ modes addressed by the receiver, which is designed to support a transmission rate equivalent to $\tilde{r}$ single-mode channels. Note that when $r < (m_t + m_r - m)$, $\tilde{r} = 0$, $P_{out} = 0$ implying that for such rates zero outage probability is achievable. A practical scheme for achieving zero outage under these conditions is proposed and discussed in [11]. In Fig. 2 we show an exemplary calculation of the outage probability as a function of the normalized rate r. These curves, obtained from our analysis were plotted in the same form as the numerical results reported in [3], except that here we assumed a fiber supporting $m = 6$ scalar modes. Note how the outage probability abruptly drops to 0 whenever r becomes smaller than $m_t + m_r - m$.

To conclude, we have studied the under-addressed optical MIMO channel, where the number of fiber modes addressed by the transmitter and receiver is allowed to be smaller than the overall number of modes existing in the fiber. This scenario was motivated by the idea of installing fibers that admit more modes that can be MIMO processed with currently available technology, in order to achieve future-proof operation [3]. While the price of not detecting all modes is notable, certain tradeoffs exist and can be used to one's advantage. Most importantly, we have shown that a performance equivalent of $m_t + m_r - m$ uncoupled single-mode channels can be achieved in all cases.

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